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STUDY OF STABILITY OF MULTIVARIABLE MULTIRATE DISCRETE CONTROL SYSTEMS

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Abstract. Study of the stability of discrete automatic control systems with standard operation modes of pulsed elements can be performed quite simply by using classical methods. However, the study of the stability of systems with complex modes of operation of pulsed elements is accompanied by certain difficulties. This statement is true primarily regarding multivariable multirate automatic control systems. This article presents an efficient algorithm for studying the stability of multivariable multirate systems based on the use of dynamic graphs. An illustration of the application of the algorithm is given.

Keywords: Discrete system, multirate system, multivariable system, dynamic graph, pulse element, stability.
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1 Introduction

The question of the systems stability analysis occupies an important place for estimating the quality of the control systems. For the discrete control systems, including multirate ones traditional methods and algebraic stability criterion were proposed based on the z-transform, the difference equations and state space method (Tou, 1971; Hiroshi et al., 1994; Gessing, 1997; Dorf & Bishop, 2001; Luchko et al., 2008; Peng et al., 2009; Shpilevaya, 2009; Vasil'ev & Malikov, 2011). In addition, the following methods were developed:

- methods and algebraic stability criterion, using a direct calculation of the entire set of roots;
- the estimating stability methods, based on the elementary matrix operations (estimating stability by norms, etc.);
- stability estimating methods, based on the use of inners, etc.

However, in some cases, traditional methods lead to cumbersome, inconvenient for the calculating equations, even for certain types of linear one-dimensional systems. For example, this is true for multirate asynchronous systems, as well as systems with a finite closure duration of pulse elements.

This article proposes to use an approach based on maximum consideration of the physical features of multirate discrete systems, namely the ability to natural decomposition and structurization. Taking into account the structural complexity of the systems, we consider the possibility of decomposing the system into a number of simpler subsystems. In this case, the problem is reduced to the study of the individual subsystems properties and the nature of the interaction of the subsystems in time.

The dynamic graph models (DGM) can be the formalized models of describing such systems. The dynamic graphs allow one the form a unified position to approach the calculation of dynamic processes and the stability of systems (Kadirova, 2010; Kadirov, 2013).

2 Dynamic graphs

Graphs with time-varying elements are the sets of vertices, edges or their weights – are called dynamic

$$G_t = \langle X_t, (V_t, \Omega_t) \rangle, \tag{1}$$

where X_t, V_t, Ω_t are accordingly, the sets of vertices, edges, and edge weights defined using the maps

$$X_t : t_* \to X,\ V_t : t_* \to V,\ \Omega_t : t_* \to \Omega,$$

$t_* = (t_1, t_2, ..., t_n)$ is a linearly ordered finite set of time moments.

In expression (1) all and individual components may change depending on the specifics of the problem being solved.

At the macro level, the systems under consideration are described using structural state graphs (SSG), reflecting the nature of the change in the structure of the system over time. The analytical description of SSG has the form as below

$$S_i = (X_i, R_i, \Omega_i), \tag{2}$$

where

$$S_i \in S = (S_1, S_2, ..., S_n),$$

$$X_i \subseteq X = (x_1, x_2, ..., x_m),$$

$$R_i \subseteq X_i \times X_i,$$

$$\Omega_i \subseteq \Omega = (\omega_1, \omega_2, ..., \omega_k).$$

Here, S_i is the structural state of system; X_i is the subset of system continuous part, match with the structural state S_i; x_j is the continuous system coordinates; R_i is binary relation in set X_i; ω_k is the edge weight v_k.

Application of the structural state graphs allows one to determine completely the dynamics of the subsystems and the nature of their interaction. In this case, the problem of pulse elements is easily solved.

To describe dynamic processes in separate structural states, we use state variable graphs (SVG). A state variable graph is an oriented weighted graph

$$G(V, \Gamma),$$

the set of vertices of which is

$$V = U(\lambda) \cup X(\lambda),$$

where $U(\lambda)$ is the subset of vertices corresponding to the input variables; $X(\lambda)$ is the subset of vertices corresponding to state variables; Γ is the mapping of V to V.

We can construct state variables graphs directly according to the type of circuit in state variables or according to the transfer function of the control object.

Thus, we can break any multirate system into m simpler subsystems (structural states), and for each of them build a subgraph. The subgraphs are connected of takes into account the fact that the state variables $\vec{x}(nT + t_i)$, calculated for the moment $t_i = t - nT$, are source nodes for the next subgraph in the interval

$$nT + t_i < t < nT + t_{i+1}.$$

Down level models, meant for the processes description within the individual subsystems, shall be set by the graphs as

$$G_t = (X'_t, X''_t, V_t), \tag{3}$$

where

$$X_t = X'_t \cup X''_t, X'_t \cap X''_t = \varnothing,$$

$$\forall x, y \in X_t \big[x, y \in V_t \Rightarrow x \in X'_t \& y \in X''_t\big],$$

$$X'_t = \big(x'_1(jT), x'_2(jT), ..., x'_k(jT)\big),$$

$$X''_t = \big(x''_1(\overline{j+1}T), x''_2(\overline{j+1}T), ..., x''_k(\overline{j+1}T)\big),$$

$$\forall (x'_i, x''_j) \in V_t \big[\Diamond x'_i, x''_j = \nabla x''_j, x'_i\big],$$

$$i, j \in J = \{1, 2, ..., k\},$$

$\nabla x''_j, x'_i \Leftrightarrow$ is a graph transmissions between the nodes $(\cdot,\cdot)$; $\Diamond x'_i, x''_j \Leftrightarrow$ is the edge weight $(\cdot,\cdot)$.

3 Algorithm for determination of stability

For multivariable multirate systems with modulation of the first or second kind, it is possible to construct a general algorithm for studying stability based on the use of dynamic graph models.

Consider a linear control system with a dynamic structure (SDS), with sequential correspondences

$$G_{jt} = (Q_t, X_t^{j-1}, X_t^j), j = 1, 2, ..., k, \tag{4}$$

where

$$X_t^0 = (x_1(0), x_2(0), ..., x_n(0)),$$

$$X_t^k = (x_1(kT), x_2(kT), ..., x_n(kT))$$

are the sets corresponding to state variables of the nth order linear dynamic system; Q is a graph of correspondence. We put to each pair $(x_i(\overline{j-1}T), x_r(jT)$ in correspondence a coefficient $a_{ri}(T)$, $i, r = 1, 2, ..., n; j = 1, 2, ..., k$. The weighted graph of the considered sequential correspondence, provided that $a_{ri}(T)$ are the values of the pulse transient functions of the channels, the signal transmission between the vertices $(x_i(\overline{j-1}T)$ and $, x_r(jT)$, will determine the state variables graph of the system with a dynamic structure. Since we can interpret each of the correspondences as an analytical description of a bipartite graph, the state variable graph is a union of bipartite graphs (Figure 1).

As already noted, depending on the specificity of the SDS, there may be several elementary structural states within the intervals $[(\overline{j-1}T), (jT)]$. We build the initial SVG, taking into account all the elementary structural states, set the cycle of their recurrence, and take the cycle time for the period. Excluding intermediate vertices corresponding to the elementary structural states, we obtain the basic structure of the SVG as a union of bipartite graphs. Moreover, the jth graph is defined on the sets of vertices $\{x^{j-1}\}, \{x^j\}$.

We can determine the stability of the SDS by the SVG of the system. The stability conditions are formulated as follows.

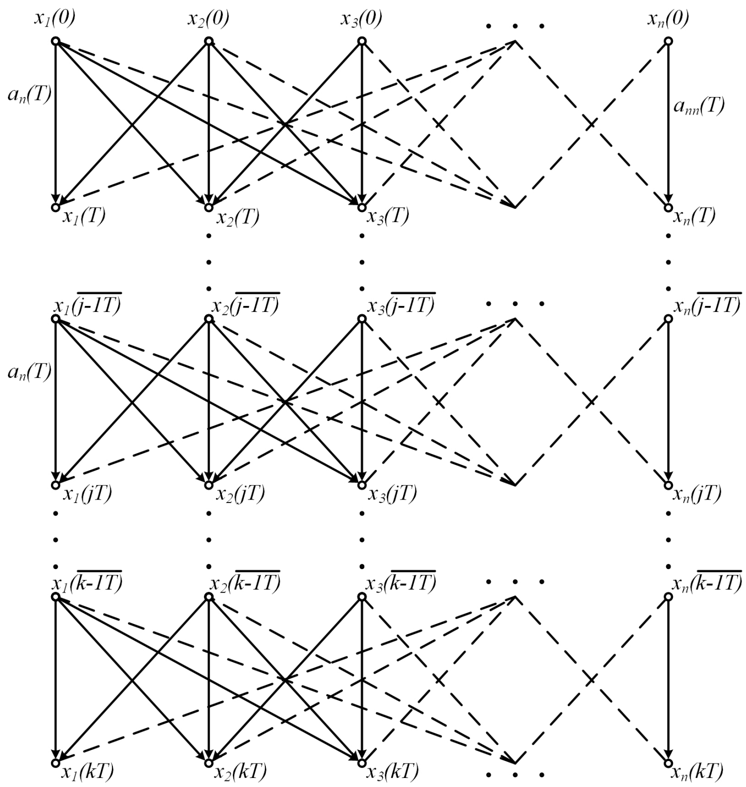

Figure 1: Dynamic graph model for determining system stability

Proposition 1. *For asymptotic stability of the linear SDS, it is necessary and sufficient that for some j, where j is the index of bipartite graph specified on the set of vertices $\{x^{j-1}\}$, $\{x^j\}$, the maximal value from the modules sums of the edges transmission coefficients, incident to the vertices from sets $\{x_i(\overline{j-1}T\}$ or $\{x_r(jT)\}$, was less than one. These sets correspond to the system state variables at $(j$–$1)$th and jth cycles of its operation.*

Proof. Sufficiency. For proof, consider a generalized adjacency matrix of the bipartite graph

$$\vec{A} = \begin{pmatrix} x_1(\overline{j-1}T) & x_2(\overline{j-1}T) & . & x_n(\overline{j-1}T) & \\ a_{11}(T) & a_{12}(T) & . & a_{1n}(T) & x_1(jT) \\ a_{21}(T) & a_{22}(T) & . & a_{2n}(T) & x_2(jT) \\ . & . & . & . & . \\ a_{n1}(T) & a_{n2}(T) & . & a_{nn}(T) & x_n(jT) \end{pmatrix}.$$

The elements $a_{ri}(T)$ of the matrix correspond to the values of the impulse response function (IRF) calculated at the time $t = T$, where T is the recurring period of the pulse elements. In the general case, if there are N connections between the vertices $x_i(\overline{j-1}T)$ and $x_r(jT)$ of the graph, in the (ir)th cell of the adjacency matrix there will be the sum of the form

$$\sum = a^1_{ri}(T) + a^2_{ri}(T) + ... + a^N_{ri}(T).$$

An analysis of Figure 1 shows that we can determine the dynamic process, defined by the bipartite graph, from the relation

$$\vec{x}(jT) = \vec{A}\vec{x}(\overline{j-1}T), \tag{5}$$

where

$$\vec{x}(jT) = (x_1(jT), x_2(jT), ..., x_n(jT))',$$

$$\vec{x}(\overline{j-1}T) = [x_1(\overline{j-1}T), x_2(\overline{j-1}T), ..., x_n(\overline{j-1}T))]'.$$

Here stands for transposition.

Similarly, excluding intermediate nodes, we obtain the dynamic process defined by the SVG on the interval $(0, kT)$, in the form

$$\vec{x}(kT) = \vec{A}^k \vec{x}(0). \tag{6}$$

Consider the condition under which $\vec{x}(kT) \to \vec{0}$ at $k \to \infty$ and $\vec{x}(0) \neq \vec{0}$. Here $\vec{x}(kT)$ is the system state vectors. From expression (6) it is clear that this is the case if $\vec{A^k} \to \vec{0}$ at $k \to \infty$. In turn, this is possible if the norms of the matrices satisfy $\left\| \vec{A^i} \right\| < 1$ for any of the values $i = 1, 2, ..., k$. If we consider that the maximal value from the modules sums of the edges transmission coefficients, incident to the vertices from the sets $\{x_i(\overline{j-1}T\}$ and $\{x_r(jT)\}$, coincides with the norms of the matrix $\vec{A}$ of the form

$$\left\| \vec{A} \right\|_I = \max_r \sum_{i=1}^{n} |a_{ri}(T)|, \tag{7}$$

$$\left\| \vec{A} \right\|_{II} = \max_i \sum_{r=1}^{n} |a_{ri}(T)|, \tag{8}$$

then sufficiency is proven.

Necessity. Suppose that $\vec{x}(0) \neq \vec{0}$, and the system is stable (asymptotically), but $\left\| \vec{A^i} \right\| > 1$. In this case, the state vector will grow unlimitedly, i.e. $\vec{x}(kT) \to \infty$ at $k \to \infty$, which contradicts the assumption. □

Algorithm 1

1. Build SVG by the successive unfolding of bipartite graphs for intervals $i = 1, 2, ..., k$.
2. Calculate each step the maximal value from the modules sums of the edges transmission coefficients, incident to the vertices $\{x_i(\overline{j-1}T\}$ and $\{x_r(jT)\}$, i.e.

$$\max_r \sum_{i=1}^{n} |a_{ri}(T)|, \max_i \sum_{r=1}^{n} |a_{ri}(T)|.$$

3. If at some step one of the maximal value of the modules sums of the edges transmission coefficients becomes less than one, then the system is stable.

4 Illustrative examples

Example 1. To illustrate of the above algorithm application we give the example of the two-variable non-phase system presented in Figure 2.

$$f^1(T) = f^2(T) = 1(T); T_1 = T_2 = 1s.$$

The system at the intervals $jT-(j+1)T$ is discretized into four structural states S_1, S_2, S_3, S_4. In Figure 3 we present the state variables graph for one of the structural states (S_1). Excluding $f^1(0), f^2(0)$, and determining the transmissions between the corresponding vertices of the graph, we obtain the first bipartite subgraph for the time interval $(0, 0^+)$. The subgraph edges transfers are the originals of transfer functions of separate and cross channels of the system, calculated for the interval $(0, 0^+)$. Similarly, we construct the bipartite subgraphs for the intervals

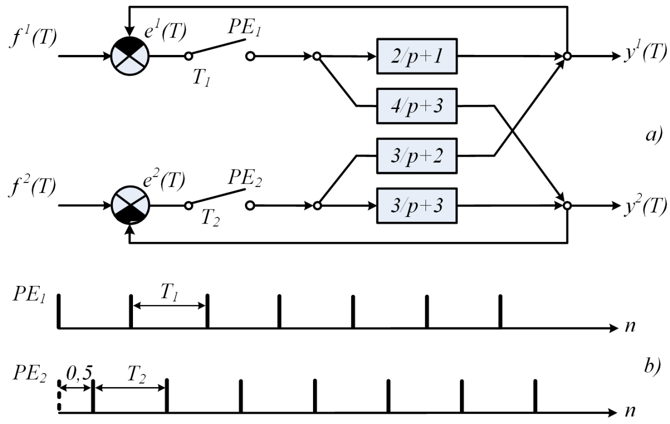

Figure 2: The structure of a system (a), the mode of operation of pulsed elements (b)

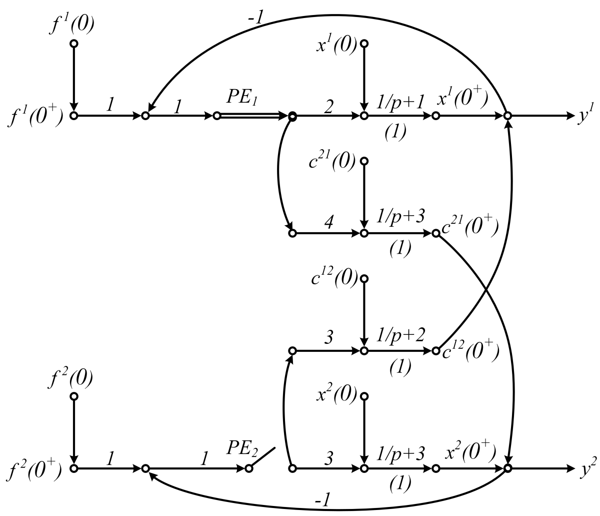

Figure 3: SVG for S_1 structural state

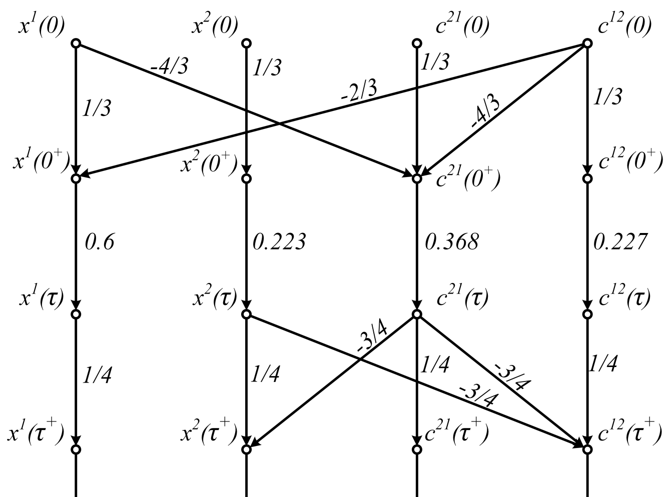

Figure 4: Bipartite graph of the system

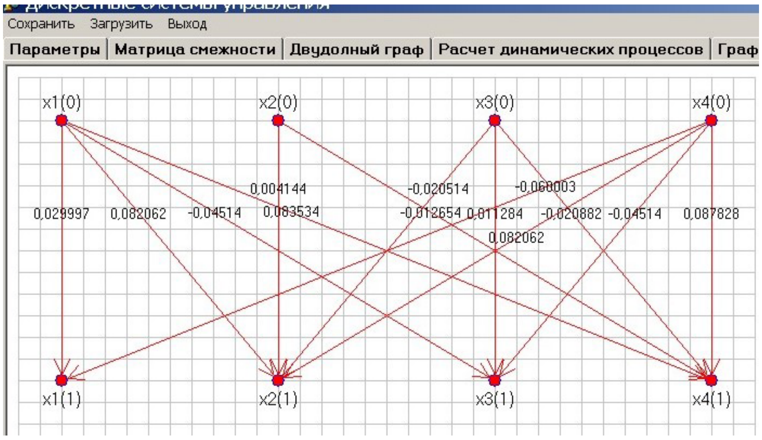

Figure 5: Transformed bipartite graph to determine system stability

$(0^+, \tau), (\tau, \tau^+), (\tau^+, T)$ (Figure 4). We transform the bipartite graph to determine the stability of the system (Figure 5).

The considered system is stable, since

$$\max_r \sum_{i=1}^{n} |a_{ri}(T)| = 0.205 < 1,$$

$$\max_i \sum_{r=1}^{n} |a_{ri}(T)| = 0.275 < 1.$$

The nature of the transients at the outputs (Figure 6) also shows the stability of the system.

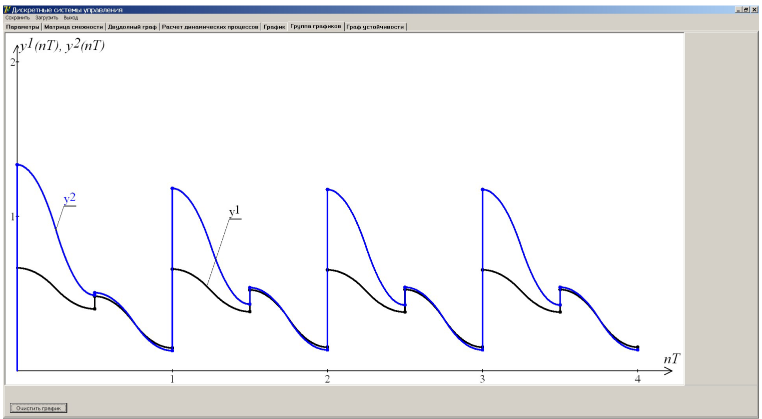

Figure 6: Transients in the system

Example 2. Let it is required to determine the stability of the two-variable non-phase system with a finite short circuit duration of the pulse elements (Figure 7) with the following parameters of the pulse elements and the continuous part

$$f_1(T) = f_2(T) = 1(T); T_1 = T_2 = 1s;$$

$$A^1(p) = 2/(p+1); A^2(p) = 4/(p+3); C^{12}(p) = 2/(p+4); C^{21}(p) = 3/(p+2);$$

$$\tau_1 = 0.2s; \tau_2 = 0.3s; t_*^1 = \{t_1, t_3, t_5, ...\}; t_*^2 = \{t_2, t_4, t_6, ...\}.$$

The pulses displacement at the PE_2 output in relation to the PE_1 output pulses is 0.1s. From the standpoint of SDS, we can represent the system as a set of 3 structural states $S_t = \{S_1, S_2, S_3\}$, (Figure 7b).

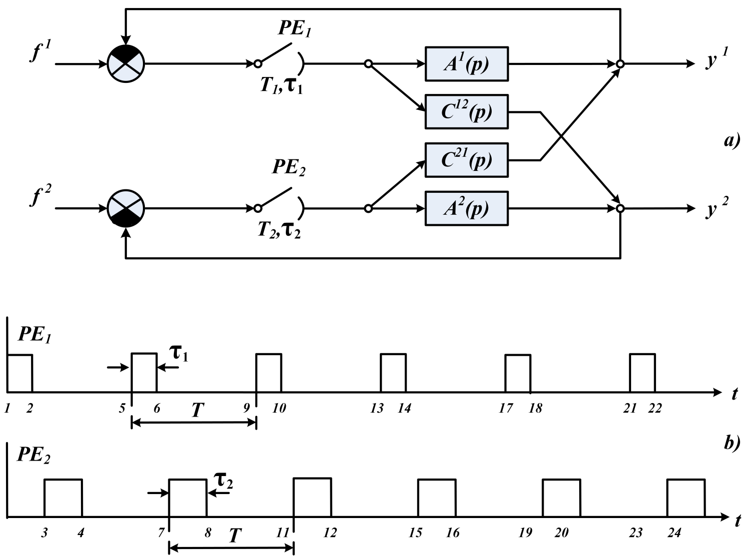

Figure 7: The structure of a system (a), the mode of operation of pulsed elements (b)

Structural state S_1 corresponds to the closed state of PE_1. Structural state S_2 is the case when both keys are open. Structural state S_3 corresponds to the closed state of PE_2. On the basis of the timing diagram of the PE work, we define the cycle of structural states and take the cycle time as a period.

We build the state variables graph and transform it into the form of a bipartite dynamic graph in the time domain. Applying the Mason formula and the inverse Laplace transform, we obtain the transmission coefficients of edges of the graph shown in Figure 8. The graph shown in Figure 8 is a model for calculating transients in the system.

To determine the stability of the system, it is necessary to exclude vertices $f^1(jT), f^2(jT)$ and outgoing edges. The corresponding bipartite graph for determining stability is shown in Figure 9.

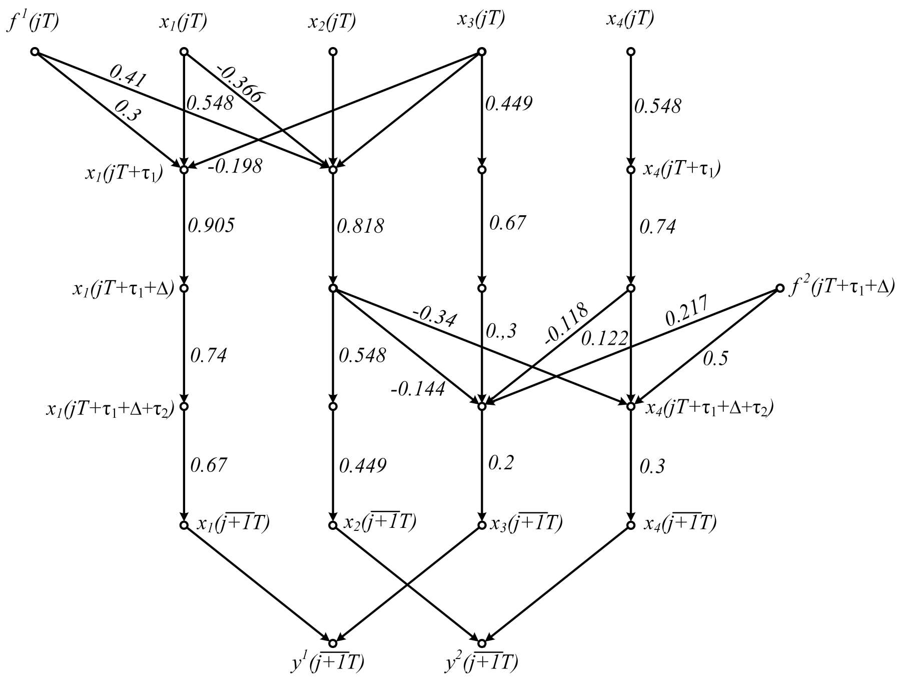

Figure 8: Bipartite graph of the system

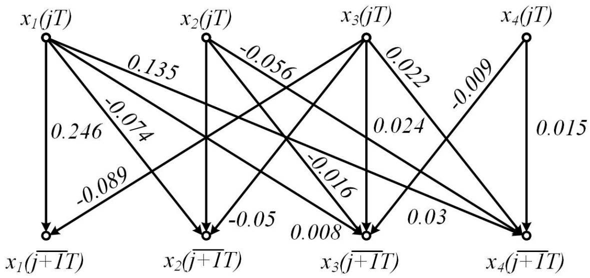

Figure 9: Transformed bipartite graph to determine system stability

Values

$$\max_{r} \sum_{i=1}^{n} |a_{ri}(T)| = 0.335 < 1,$$

$$\max_{i} \sum_{r=1}^{n} |a_{ri}(T)| = 0.359 < 1.$$

indicate that the system is stable. Transient curves y^1, y^2 (Figure 10) also confirm the stability of the system.

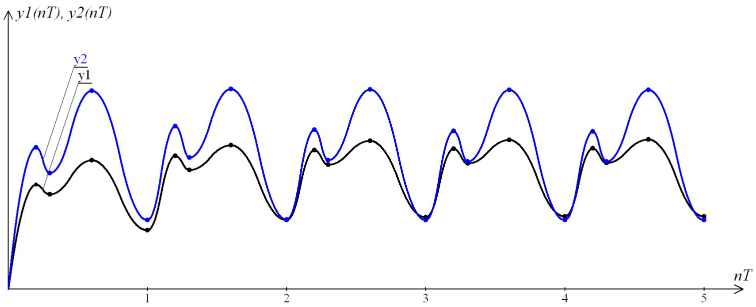

Figure 10: Transients in the system

Note that when applying traditional methods, usually fundamental difficulties are met in formalizing the complex mode of operation of PE and solving the problem of the high dimensionality of the systems. Therefore, traditional methods are practically inapplicable to multivariable systems with a finite closure duration of PE.

5 Conclusion

In conclusion, we note that the article proposed an effective method for studying the stability of structurally and parametrically complex discrete automatic control systems. The dynamic graphs method covers a wide class of multirate control systems. We showed the operation of the algorithm by examples of a two-variable non-phase system and a two-variable system with modulation of the second kind. An important feature of the proposed mathematical graph models is that on their basis, it is possible to study both the stability and the dynamics of the system's functioning.

6 Acknowledgement

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